<u>*Representation of Signals & Systems*</u>

Reference: Chapter 2,Communication Systems, Simon Haykin

<u>*1. Hilbert Transform*</u>

- Fourier transform $\Rightarrow$ frequency content of a signal (frequency selectivity – designing frequency-selective filters for the separation of signals on the basis of frequency contents)

Here we use *phase selectivity* $\Rightarrow$ uses phase shifts between signals under consideration to achieve separation.

- Simplest phase shift is 180^0 $\Rightarrow$ polarity reversal in the case of a sinusoidal signal. Requires the use of an *ideal transformer* in general.

- Another shift of interest is $\pm\, 90^0$ $\Rightarrow$ the resulting *function of **time*** is known as the Hilbert transform of the signal.

Consider a signal *g(t)* with Fourier transform *G(f)*.
The <u>*Hilbert transform*</u> of *g(t)* is defined by

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(\tau)}{t-\tau}d\tau = g(t) * \frac{1}{\pi t} \quad , \ \text{*- convolution} \quad (1)$$

This is a linear operation. The <u>*inverse Hilbert transform*</u> is defined by

$$g(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\hat{g}(\tau)}{t-\tau}d\tau = \hat{g}(t) * -\frac{1}{\pi t} \qquad (2)$$

The functions *g(t) and* $\hat{g}(t)$ are said to constitute a *Hilbert-transform pair*.

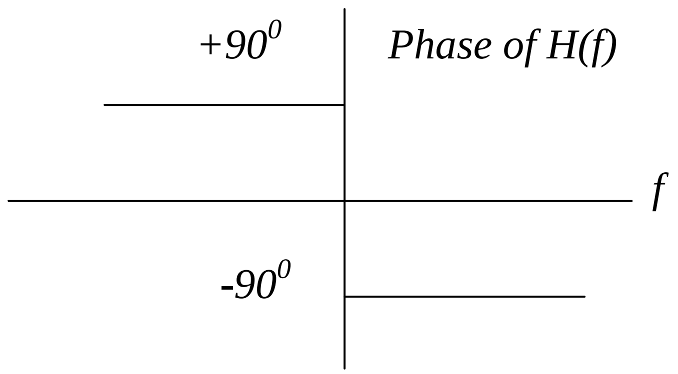
In terms of Fourier transform;

$$\frac{1}{\pi t} \Leftrightarrow -j\ sign(f)$$

Thus $$\hat{G}(f) = -j\,\mathrm{sgn}(f)G(f)$$

So for a given signal *g(t)* we may obtain its Hilbert transform $\hat{g}(t)$ through a two port device with transfer function *-j sign(f)*. It may be considered as giving -90^0 phase shift for all positive frequencies and $+90^0$ for all negative frequencies.

The amplitudes of all frequency components in the signal are unaffected. Such a device is called as a *Hilbert transformer*.



Hilbert transform is used to:

- *realize phase selectivity in **SSB modulation**.*
- *provide the mathematical basis for the representation of **band-pass signals**.*

The Hilbert transform as defined above applies to any signal that is Fourier transformable. Thus it may be applied to *energy* signals as well as *power* signals.

e.g. ***Sinusoidal functions***

Consider $g(t) = \cos(2\pi f_c t)$

$G(f) = 1/2. [\delta(f-f_c) + \delta(f+f_c)]$

Therefore $\hat{G}(f) = -j\, sgn(f).G(f)$

$= -j/2. [\delta(f-f_c) + \delta(f+f_c)] sgn(f)$

$= 1/2j. [\delta(f-f_c) - \delta(f+f_c)]$

This is the Fourier transform of $\sin(2\pi f_c t)$. *So the H.T. of the cosine function is* $\sin(2\pi f_c t)$. *Similarly* $\sin(2\pi f_c t)$ *has a H.T. equal to* $-\cos(2\pi f_c t)$.

Properties of the Hilbert Transform

Hilbert transform *differs* from the Fourier transform as it operates only in *time domain.* The signal is usually assumed to be *real valued*.

1. *A signal g(t) and its Hilbert transform have the same amplitude spectrum.* $|-j\, sgn(f)| = 1 \ \forall f$.
2. *If* $\hat{g}(t)$ *is the H.T. of g(t), then the H.T. of* $\hat{g}(t)$ *is –g(t).* ***(check !!)***
3. *A signal g(t) and its H.T. are orthogonal.*

   Here we use multiplication theorem in F.T.

$$\int_{-\infty}^{\infty} g(t)\hat{g}(t)dt = \int_{-\infty}^{\infty} G(f)\hat{G}(-f)df$$

$$= j\int_{-\infty}^{\infty} G(f)\hat{G}(-f)df$$

$$= j\int_{-\infty}^{\infty}\operatorname{sgn}(f)G(f)G(-f)df$$

$$= j\int_{-\infty}^{\infty}\operatorname{sgn}(f)G(f)G^*(f)df$$

$$= j\int_{-\infty}^{\infty}\operatorname{sgn}(f)\left|G(f)\right|^2 df$$

where G(-f) = G(f) for g(t) real. The integrand is odd.*

Thus $\int_{-\infty}^{\infty} g(t)\hat{g}(t)dt = 0.$

Similarly we may show that a *power* signal *g(t)* and its Hilbert transform $\hat{g}(t)$ are orthogonal over one period;

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(t)\hat{g}(t)dt = 0.$$

(check the earlier example)

2. Pre-Envelope

Consider a real-valued signal *g(t)*. We define the *pre-envelope* of the signal *g(t)* as the complex-valued function

$$g_+(t) = g(t) + j\,\hat{g}(t)$$

where $\hat{g}(t)$ is the H.T. of *g(t)*.

We note that the given signal *g(t)* is the real part of the pre-envelope $g_+(t)$ and the H.T. of the signal is the imaginary part of the pre-envelope.

Pre-envelope makes the handling of band-pass signals and systems easier.

The Fourier transform:

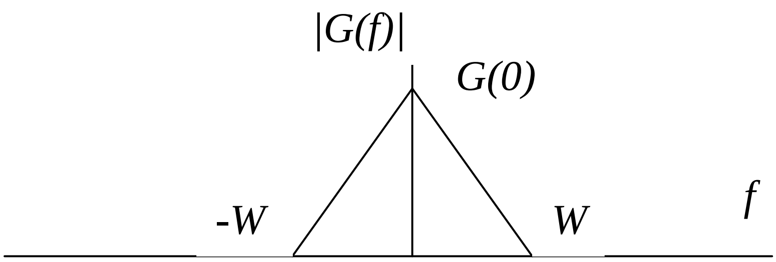
$$G_+(f) = G(f) + j\,[-j\operatorname{sgn}(f)]G(f)$$

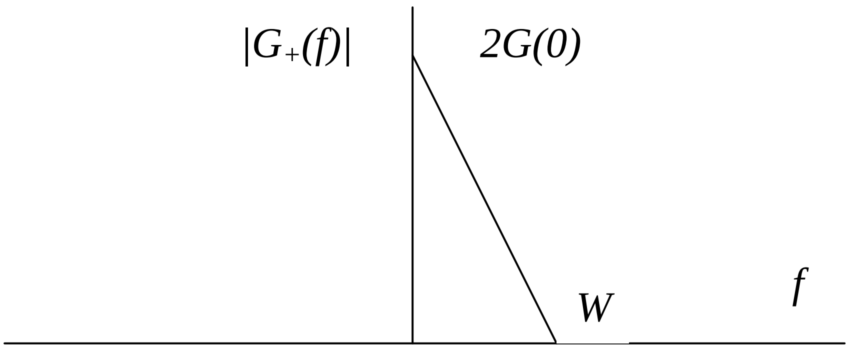
Thus

$$G_+(f) = \begin{cases} 2G(f), & f > 0 \\ G(0), & f = 0 \\ 0, & f < 0 \end{cases}$$

Where *G(0)* is the value of *G(f)* at frequency *f=0*.

Therefore $g_+(t) = 2\int_0^{\infty} G(f)\exp(j2\pi ft)\,df$





In the above case we used a *low-pass signal* but pre-envelope can be defined for any signal with a spectrum.

We may define the pre-envelope for *negative frequencies* as

$$g_{-}(t) = g(t) - j\,\hat{g}(t)$$

The two pre-envelopes $g_{+}(t)$ and $g_{-}(t)$ are complex conjugates of each other.

$$G_{-}(f) = \begin{cases} 0, & f > 0 \\ G(0), & f = 0 \\ 2G(f), & f < 0 \end{cases}$$

3. Canonical Representation of Band-Pass Signals

We say that a signal *g(t)* is a *band-pass signal* if its Fourier transform *G(f)* is *non-negligible* only in a band of frequencies of total extent *2W*, e.g, centered about some frequency $\pm f_c$.

We refer to f_c as carrier frequency. In most cases *2W* is small compared with $f_c \Rightarrow$ *narrow-band signal.*

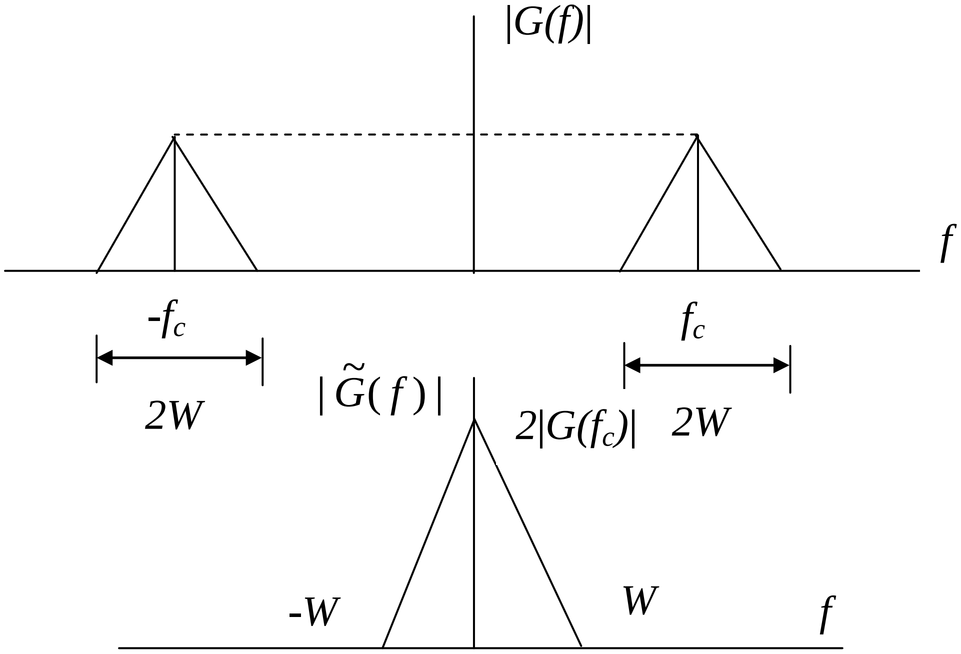
Let the pre-envelope of such a signal be expressed as

$$g_+(t) = \tilde{g}(t)\exp(j2\pi f_c t)$$

We refer to $\tilde{g}(t)$ as the *complex envelope* of the signal. The spectrum of $g_+(t)$ is limited to the band $f_c - W \le f \le f_c + W$.

We find that the spectrum of $\tilde{g}(t)$ is therefore limited to the band $-W \le f \le W$ and centered at the origin.

$\Rightarrow$ The complex envelope $\tilde{g}(t)$ of a band-pass signal *g(t)* is a *low-pass signal.*



$$g(t) = Re[g_+(t)]=Re\,[\tilde{g}(t)\,exp(j2\pi f_c t)]$$

In general, $\tilde{g}(t)$ is complex. To emphasize this we usually express it as

$$\tilde{g}(t) = g_I(t) + j\,g_Q(t)$$

where $g_I(t)$ and $g_Q(t)$ are both real-valued low-pass functions.

The low-pass property is *inherited* from the complex envelope $\tilde{g}(t)$.

Therefore we can express the original band-pass signal in the *canonical form* as

$$g(t) = g_I(t)cos(2\pi f_c t) - g_Q(t)\,sin(2\pi f_c t)$$

We refer to $g_I(t)$ as the *in-phase component* of the band-pass signal $g(t)$ and to $g_Q(t)$ as the *quadrature component* of the signal.

We can now give the corresponding *phasor* diagrams.

Also the following schemes can be given to obtain $g_I(t)$ and $g_Q(t)$ from $g(t)$ and vice-versa.

The multiplication of low pass in phase component $g_I(t)$ by $cos(2\pi f_c t)$ and the multiplication of low pass quadrature component $g_Q(t)$ by $sin(2\pi f_c t)$ represent *linear forms of modulation.*

Given that the carrier frequency f_c is sufficiently large, $g(t)$ is referred to as a *pass-band signaling waveform.*

Correspondingly the mapping from $g_I(t)$ and $g_Q(t)$ into $g(t)$ is known as *passband modulation.*

We can express $\tilde{g}(t)$ in polar form as

$$\tilde{g}(t) = a(t) \exp [j\phi(t)]$$

where *a(t)* and *ϕ(t)* are both real-valued low-pass functions.

Based on this polar representation, the original band-pass signal *g(t)* is defined by

$$g(t) = a(t)\cos [2\pi f_c t + \phi(t)]$$

We refer to *a(t)* as the *natural envelope* or just *envelope* of bandpass signal *g(t)* and to *ϕ(t)* as the *phase* of the signal.

The above equation represents a *hybrid form of amplitude modulation and angle modulation.*

- *The information content of the signal g(t) is completely represented by the complex envelope* $\tilde{g}(t)$.

4 Band-Pass Systems

Already know how to represent band-pass signals.

$\Rightarrow$ analyze band-pass systems based on the relation between low-pass and band-pass systems. (*Hilbert transform)*

Consider a *narrow-band* signal $x(t)$ with Fourier transform $X(f)$.

Assume that the $X(f)$ is limited to $\pm W$ *Hz* of the carrier frequency f_c. Also assume that $W < f_c$.

Let $$x(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)$$

Where $x_I(t)$ in-phase component, $x_Q(t)$ quadrature component.

The complex envelope of $x(t)$,

$$\tilde{x}(t) = x_I(t) + jx_Q(t)$$

Let $x(t)$ be applied to a LTI (linear time-invariant) band-pass system with impulse response $h(t)$ and transfer function $H(f)$.$H(f)$ is limited to $\pm B$ *Hz* of the carrier frequency f_c.

The system bandwidth $2B$ is either narrower or equal to the i/p signal bandwidth $2W$.

We want to express $h(t)$ in terms of $h_I(t)$ and $h_Q(t)$ its in-phase and quadrature components.

Thus, $$h(t) = h_I(t)\cos(2\pi f_c t) - h_Q(t)\sin(2\pi f_c t)$$

Define the *complex impulse response* of the band-pass system as

$$\tilde{h}(t) = h_I(t) + jh_Q(t)$$

So $\quad h(t) = Re[\tilde{h}(t)\exp(j2\pi f_c t)]$

Note that $h_I(t)$, $h_Q(t)$ and $\tilde{h}(t)$ are all low-pass functions limited to $-B \le f \le B$.

$$2h(t) = \tilde{h}(t)\exp(j2\pi f_c t) + \tilde{h}^*(t)\exp(-j2\pi f_c t)$$

$\tilde{h}^*(t)$ is the complex conjugate of $\tilde{h}(t)$.

Taking Fourier transform,

$$2H(f) = \tilde{H}(f - f_c) + \tilde{H}^*(-f - f_c)$$

Now for a real impulse response $h(t)$, $H^*(f) = H(-f)$.

As $\tilde{H}(f)$ represents a low pass function limited to $|f| \le B$ with $B < f_c$ we can obtain from the above relation:

$$\tilde{H}(f - f_c) = 2H(f) \qquad f > 0$$

So we can find $\tilde{H}(f)$ by taking the positive frequency part of $H(f)$ and shifting it to origin scaled by 2.

Then taking the inverse Fourier transform of $\tilde{H}(f)$ we can find the complex impulse response $\tilde{h}(t)$.

Without loss of generality we assume that $X(f)$ and $H(f)$ are both centered around f_c. Let $y(t)$ be the output signal. It is also a band-pass signal. Hence,

$$y(t) = Re[\tilde{y}(t)\exp(j2\pi f_c t)]$$

Also $\quad y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$

In terms of pre-envelopes, we have

$h(t) = \mathrm{Re}\,[h_+(t)]$ $\qquad$ $x(t) = \mathrm{Re}\,[x_+(t)]$

Therefore $\qquad y(t) = \displaystyle\int_{-\infty}^{\infty} \mathrm{Re}[h_+(\tau)]\,\mathrm{Re}[x_+(t-\tau)]\,d\tau$

It can be shown that

$$\int_{-\infty}^{\infty} \mathrm{Re}[h_+(\tau)]\,\mathrm{Re}[x_+(\tau)]\,d\tau = \frac{1}{2}\mathrm{Re}\left[\int_{-\infty}^{\infty} h_+(\tau)\,x_+^*(\tau)\,d\tau\right]$$

As we use $x(-\tau)$ instead of $x(\tau)$ we can avoid the conjugate. Thus

$$y(t) = \frac{1}{2}\mathrm{Re}\left[\int_{-\infty}^{\infty} h_+(\tau)\,x_+(t-\tau)\,d\tau\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\int_{-\infty}^{\infty} \tilde{h}(\tau)\exp(j2\pi f_c\tau)\,\tilde{x}(t-\tau)\exp(j2\pi f_c(t-\tau))\,d\tau\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\exp(j2\pi f_c t)\int_{-\infty}^{\infty} \tilde{h}(\tau)\,\tilde{x}(t-\tau)\,d\tau\right]$$

Therefore,

$$2\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{h}(\tau)\,\tilde{x}(t-\tau)\,d\tau = \tilde{h}(t) * \tilde{x}(t)$$

$$= [\,h_I(t) + j\,h_Q(t)] * [\,x_I(t) + j\,x_Q(t)\,]$$

If $\tilde{y}(t) = y_I(t) + j\, y_Q(t)$

Then $$2y_I(t) = h_I(t) * x_I(t) - h_Q(t) * x_Q(t)$$

$$2y_Q(t) = h_Q(t) * x_I(t) + h_I(t) * x_Q(t)$$

Therefore, for the purpose of obtaining the in-phase and quadrature components of the complex envelope $\tilde{y}(t)$ of the system output, we can use the <u>*low-pass equivalent model*</u> shown below.