Algorithms for computing a planar homography from conics in correspondence

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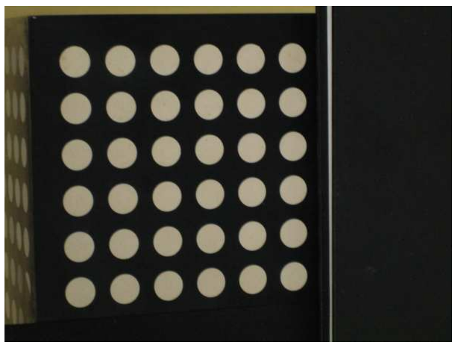
Introduction

This presentation deals with the computation of a planar homography from conic correspondences.



$$\xrightarrow{\mathbf{H}\ ?}$$

Two new algorithms are proposed:

- a linear SVD-based algorithm for the case of three or more correspondences
- another algorithm for the minimal case of two correspondences

Both algorithms use only linear algebra and are easy to implement.

Background

The transformation between two perspective views of a plane is a homography.

- Most often the homography is computed from point correspondences.
- Also conics have been used

  [1] Sugimoto: A linear algorithm for computing the homography from conics in correspondence. *Journal of Mathematical Imaging and Vision*, 2000.

  [2] Mudigonda et. al: Geometric structure computation from conics. ICVGIP, 2004.

Background

The transformation between two perspective views of a plane is a homography.

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  [1] Sugimoto: A linear algorithm for computing the homography from conics in correspondence. *Journal of Mathematical Imaging and Vision*, 2000.

  [2] Mudigonda et. al: Geometric structure computation from conics. ICVGIP, 2004.
- The proposed SVD-based algorithm requires only 3 correspondences while that in [1] requires 7.
- The proposed algorithm for 2 correspondences does not require solving polynomial equations as that in [2].

Properties of conics

In homogeneous coordinates, a general conic section $\mathbf{C}$ is represented with a real symmetric 3×3 matrix,

$$\mathbf{x}^\top \mathbf{C}\mathbf{x} = 0.$$

Under the point homography $\mathbf{x}' = \mathbf{H}\mathbf{x}$ a conic $\mathbf{C}$ transforms to $\mathbf{C}' = \mathbf{H}^{-\top}\mathbf{C}\mathbf{H}^{-1}$.

Due to undetermined scale the correspondence constraint

$$\mathbf{C}' \sim \mathbf{H}^{-\top}\mathbf{C}\mathbf{H}^{-1}$$

provides 5 scalar constraints while the homography has 8 degrees of freedom. Hence, at least two correspondences are required.

General case: n *correspondences*

Assume that we have the conic correspondences $\mathbf{C}_i \leftrightarrow \mathbf{C}'_i$, between two-planes related by a homography $\mathbf{H}$. Further, assume that $\det \mathbf{C}_i \neq 0$, $\det \mathbf{C}'_i \neq 0$, $\det \mathbf{H} \neq 0$.

By fixing $\det \mathbf{H} = 1$ and normalizing the conics so that $\det \mathbf{C}_i = \det \mathbf{C}'_i$ we get

$$\mathbf{C}_i = \mathbf{H}^\top \mathbf{C}'_i \mathbf{H}, \quad i = 1, \dots, n$$

Any two of the above equations gives

$$\mathbf{C}_i^{-1} \mathbf{C}_j = \mathbf{H}^{-1} \mathbf{C}'^{-1}_i \mathbf{C}'_j \mathbf{H}.$$

This can be written in the form

$$\mathbf{C}'^{-1}_i \mathbf{C}'_j \mathbf{H} - \mathbf{H} \mathbf{C}_i^{-1} \mathbf{C}_j = \mathbf{0}$$

which is a set of linear equations in the elements of $\mathbf{H}$.

General case: n correspondences

The linear equations can be written in the form

$$\mathbf{M}_{ij}\mathbf{h} = \mathbf{0},$$

where $\mathbf{h}$ is a 9×1 vector containing the elements of $\mathbf{H}$ and $\mathbf{M}_{ij}$ is a 9×9 matrix determined by the conics.

By considering all ordered pairs among the n conics we get an overdetermined set of $9n(n-1)$ equations

$$\mathbf{M}\mathbf{h} = \mathbf{0}$$

so that the null space is usually one dimensional when $n \geq 3$.

In practice, the solution minimizing $||\mathbf{M}\mathbf{h}||$ with $||\mathbf{h}|| = 1$ is obtained by SVD. If desired, one may scale the elements of $\mathbf{h}$ so that $\det \mathbf{H} = 1$.

Algorithm, $n \geq 3$

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(i) Given the conic correspondences normalize all the coefficient matrices to have unit Frobenius norm. Denote the obtained matrices by $\mathbf{C}_i$, $\mathbf{C}'_i$, $i = 1, \ldots, n$. Then, for all i, replace $\mathbf{C}_i$ with $s_i \mathbf{C}_i$ where $s_i = \left(\frac{\det \mathbf{C}'_i}{\det \mathbf{C}_i} \right)^{1/3}$.

(ii) For each ordered pair $\{i, j\}$, $i, j = 1, \ldots, n$, compute $\mathbf{C}_i^{-1} \mathbf{C}_j$ and $\mathbf{C}_i'^{-1} \mathbf{C}'_j$ and use these to form the matrices $\mathbf{M}_{ij}$.

(iii) Stack all the matrices $\mathbf{M}_{ij}$ to a $9n(n-1) \times 9$ matrix $\mathbf{M}$ and compute its singular value decomposition $\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^\top$ so that the singular values are in descending order on the diagonal of $\mathbf{S}$. Set $\mathbf{h}$ to be the last column of $\mathbf{V}$.

(iv) The vector $\mathbf{h}$ contains the elements of the homography matrix $\mathbf{H}$ (if the configuration of conics is such that they determine the homography uniquely).

Minimal case: 2 *correspondences*

Lemma. *If* $\mathbf{C}$ *is real symmetric then* $\mathbf{C}=\mathbf{F}\mathbf{F}^\top$ *for some* $\mathbf{F}$. *If* $\mathbf{C}$ *is invertible and* $\mathbf{C}=\mathbf{F}\mathbf{F}^\top=\mathbf{F}_1\mathbf{F}_1^\top$, *then* $\mathbf{F}_1=\mathbf{F}\mathbf{R}$ *where* $\mathbf{R}^\top\mathbf{R}=\mathbf{I}$.

By fixing again $\det \mathbf{H} = 1$ and $\det \mathbf{C}_i = \det \mathbf{C}'_i$ we get

$$\begin{aligned} \mathbf{C}_1 &= \mathbf{H}^\top \mathbf{C}'_1 \mathbf{H} \qquad \Rightarrow \qquad \mathbf{F}_1\mathbf{F}_1^\top = \mathbf{H}^\top \mathbf{F}'_1 \mathbf{F}'^\top_1 \mathbf{H} \\ \mathbf{C}_2 &= \mathbf{H}^\top \mathbf{C}'_2 \mathbf{H}. \end{aligned}$$

Thus, $\mathbf{H} = \mathbf{F}'^{-\top}_1 \mathbf{R} \mathbf{F}_1^\top$ and substituting this to the latter equation gives a pair of equations which is equivalent to the original one:

$$\mathbf{R}^\top \mathbf{R} = \mathbf{I},$$

$$\mathbf{R}^\top \mathbf{A} \mathbf{R} = \mathbf{B}.$$

Here $\mathbf{R}$ is the unknown and $\mathbf{A}$ and $\mathbf{B}$ are known complex symmetric matrices.

Minimal case: 2 *correspondences*

The task is to solve $\mathbf{R}$ from

$$\mathbf{R}^\top \mathbf{A}\mathbf{R} = \mathbf{B}.$$

When $\mathbf{A}$ and $\mathbf{B}$ have distinct eigenvalues they are diagonalizable and the solutions are

$$\mathbf{R} = \mathbf{Q}\mathbf{P}\mathbf{U}^\top,$$

where $\mathbf{Q}$ and $\mathbf{U}$ contain the eigenvectors of $\mathbf{A}$ and $\mathbf{B}$, respectively, and

$$\mathbf{P} = \mathrm{diag}(\pm 1, \pm 1, \pm 1).$$

Only four of the above eight choices for $\mathbf{P}$ gives $\det \mathbf{H} = 1$
$\rightarrow$ 4 solutions

If $\mathbf{A}$ and $\mathbf{B}$ have a multiple eigenvalue and are diagonalizable then there are infinitely many solutions. The case that the matrices are not diagonalizable is possible but unlikely in practice.

Algorithm, $n = 2$

(i) As before, normalize the conics so that $\det \mathbf{C}_i = \det \mathbf{C}'_i$.

(ii) Compute the eigendecompositions $\mathbf{C}_1 = \mathbf{V}_1 \mathbf{D}_1 \mathbf{V}_1^\top$ and $\mathbf{C}'_1 = \mathbf{V}'_1 \mathbf{D}'_1 \mathbf{V}'^\top_1$ and set $\mathbf{F}_1 = \mathbf{V}_1 \mathbf{D}_1^{1/2}$ and $\mathbf{F}'_1 = \mathbf{V}'_1 \mathbf{D}'^{1/2}_1$.

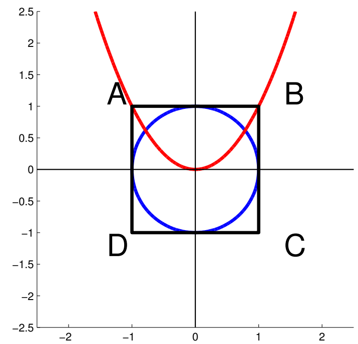
(iii) Compute $\mathbf{A} = \mathbf{F}'^{-1}_1 \mathbf{C}'_2 \mathbf{F}'^{-\top}_1$ and $\mathbf{B} = \mathbf{F}_1^{-1} \mathbf{C}_2 \mathbf{F}_1^{-\top}$ and their eigendecompositions $\mathbf{A} = \mathbf{Q} \mathbf{D}_\mathrm{A} \mathbf{Q}^\top$ and $\mathbf{B} = \mathbf{U} \mathbf{D}_\mathrm{B} \mathbf{U}^\top$ where $\mathbf{D}_\mathrm{A} = \mathrm{diag}(a_1, a_2, a_3)$ and $\mathbf{D}_\mathrm{B} = \mathrm{diag}(b_1, b_2, b_3)$.

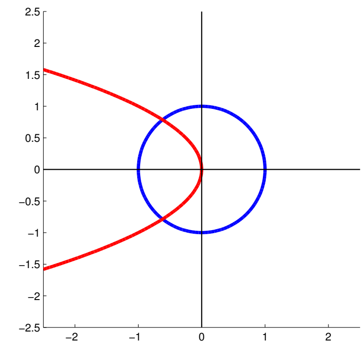
(iv) Go through all six possible permutations of the eigenvalues b_k and compute distances $\sum_k |a_k - b_k|^2$. Choose the ordering that gives the smallest distance and permute the eigenvectors correspondingly to get a new $\mathbf{U}$.

(v) Set $\mathbf{P}_1 = \mathrm{diag}(1, 1, 1)$, $\mathbf{P}_2 = \mathrm{diag}(-1, 1, 1)$, $\mathbf{P}_3 = \mathrm{diag}(1, -1, 1)$ and finally $\mathbf{P}_4 = \mathrm{diag}(1, 1, -1)$. The four solutions are $\mathbf{H}_k = \mathbf{F}'^{-\top}_1 \mathbf{Q} \mathbf{P}_k \mathbf{U}^\top \mathbf{F}_1^\top$, $k = \{1, 2, 3, 4\}$. In general, one of these solutions should be the geometrically correct one.

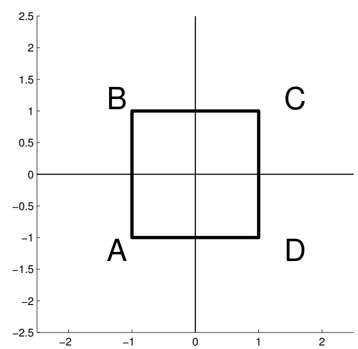
Simple example

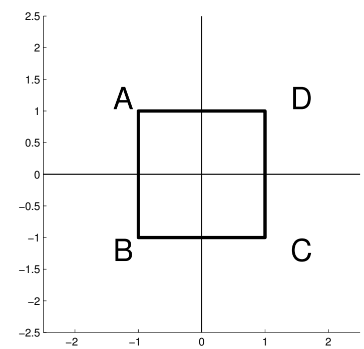
Correspondence of a parabola and a circle under rotation:





Two real solutions, rotation and rotation $+$ reflection:

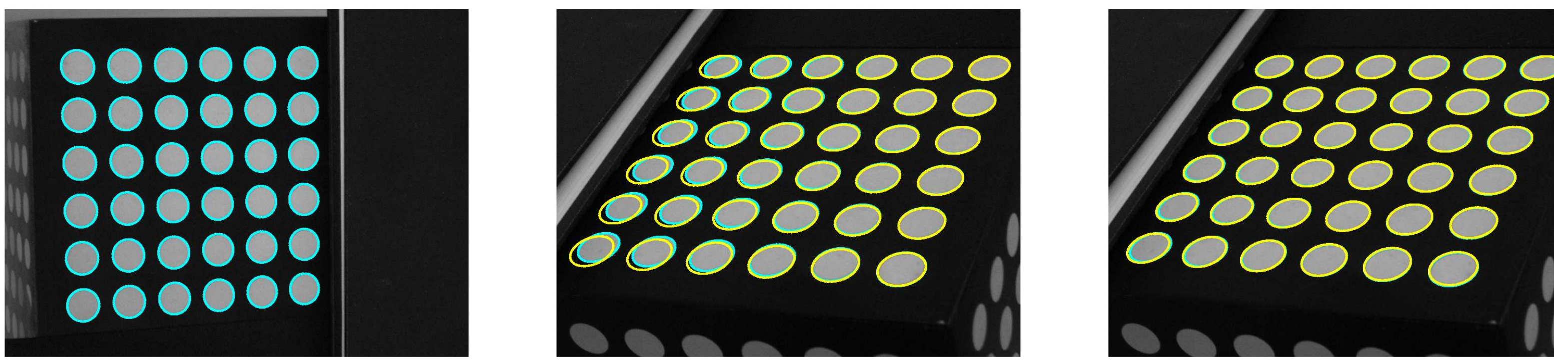




Real example

Two images of a plane containing white circles. The detected ellipses are in cyan. The estimated homography is illustrated in yellow.

Middle: 2 correspondences used
Right: 36 correspondences used



Conclusions

- Two new algorithms were proposed for computing a homography from conic correspondences.
- Both algorithms require only linear algebra and are easy to implement.
- Experiments showed that the algorithms provide a reasonable estimate also when no exact solution exists due to measurement errors in the conic coefficients.

Matlab implementations of the algorithms are available at http://www.ee.oulu.fi/∼jkannala/