

Near-Optimal SLAM Exploration in Gaussian Processes

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Abstract—In this paper we examine near-optimal SLAM exploration in Gaussian processes. We propose a submodular sensing quality function that extends studies from discrete sensor placement to an autonomous sampling scheme where sensing sites must be visited frequently. This is beneficial in the SLAM context, where sensing sites themselves bear uncertainties. Also in time-critical applications, we have to balance modeling accuracy against sensing time, which introduces noisy samples with only limited replications at each site.

Our SLAM studies were inspired by our previous studies on indoor mobile robot localization that utilizes ambient magnetic fields. Those studies were based on observations that most indoor magnetic field distortions originate from concrete structures, which make these spatial fluctuations stable over time. Magnetic vector fields can be modelled by three orthogonal Gaussian processes that provide a flexible framework for localization.

In the first part of this paper we prove that, in Gaussian processes, mutual information provides near-optimal solutions when each sensing site can be visited infinite times. We also conjecture that the optimal sites are within a set of sensing sites where the quality of processes is to be estimated. This enables near-optimal sensing when the sensing sites are defined in a probability space.

In the second part we provide technical details and equations required when working in a SLAM context. We show that, considering pose uncertainties, we can use a greedy policy to select sensing sites that provide a near-optimal solution for modeling and give information required to decrease pose uncertainties. Finally, in order to evaluate our approach, we built preliminary simulations of vector fields.

I. INTRODUCTION

SLAM exploration is usually considered in binary occupancy distributions or in discrete feature distributions (see e.g. [13], [6]). We extend these views to continuous distributions of spatial variables which are given in metric spaces by stochastic processes that express a correlation found in nature. This approach was inspired by our previous studies on indoor mobile robot localization that utilizes ambient magnetic fields [3]. Those studies were based on observations that most indoor magnetic field distortions originate from concrete structures, which make these spatial fluctuations stable over time.

One way to model these magnetic vector fields is to use three orthogonal Gaussian processes. Gaussian processes [10] provide a flexible framework for regression, requiring only a limited number of hyperparameters for successful prediction. These have been extensively studied by Plagemann [9] in robotic regression and by Guestrin *et al.* [2] in near-optimal sensing. Our approach for near-optimal SLAM

exploration extends these studies from near-optimal sensing (see e.g. [2], [7], [12]) to an application area where sensing sites themselves bear uncertainties. In the following sections we give the theoretical details of our approach.

II. NEAR-OPTIMAL SENSING

We consider a random vector $\mathbf{Y}_{\mathcal{V}}$ on $\mathbf{R}^{|\mathcal{V}|}$ where \mathcal{V} denotes a set of possible sensing sites. Following the notation of Krause [7], we can now define the sensing quality of a subset $\mathcal{A} \subset \mathcal{V}$ with a set function

$$F(\mathcal{A}) = \int p(\mathbf{y}_{\mathcal{V}})u(\mathbf{y}_{\mathcal{V}}, \mathcal{A})d\mathbf{y}_{\mathcal{V}}, \quad (1)$$

where $\mathbf{y}_{\mathcal{V}}$ is a realization of a random vector and $u(\mathbf{y}_{\mathcal{V}}, \mathcal{A})$ is a utility function.

In spatial sampling design (see e.g. [1]), the objective is to find a set of sensing sites that maximize sensing quality. In general, however, this is a *NP*-hard problem and often one must rely on a greedy algorithm to find a suboptimal solution.

Near-optimal sensing (see Guestrin *et al.* [2]) refers to solutions that can be guaranteed to attain some level of optimality. In their studies, Guestrin *et al.* proved that for a mutual information criterion, a greedy algorithm guarantees near-optimality when $|\mathcal{A}| \ll |\mathcal{V}|$. In the following subsections we prove that this can be extended to applications where we are required to take more than one sample from each possible sensing site. In addition, we propose a conjecture that allows us to consider near-optimality also in continuous sensing spaces.

A. Mutual information criterion

Near-optimal sensing originates from the studies by Guestrin *et al.* [2], where they based their research on nondecreasing submodular set functions. As was proved by Nemhauser [8], for functions of this kind, a greedy algorithm can be guaranteed to attain $1 - e^{-1}$ optimality. In sensing applications, as noted in [7], a greedy algorithm often yields even much better optimality.

Following from the work of Nemhauser *et al.* [8], submodularity in set functions can be stated as

$$F(\mathcal{A} \cup \{s\}) - F(\mathcal{A}) \geq F(\mathcal{B} \cup \{s\}) - F(\mathcal{B}), \quad \text{for all } s \in \mathcal{V} \quad (2)$$

where $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{V}$. Intuitively this means the increment provided by adding a specific element is never greater for a superset, which is closely related to concavity in vector spaces.

In spatial statistics, a widely used criterion is based on the concept of selecting sensing sites that provide minimum

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entropy for a random vector at unobserved sites [1], [11]. Resulting from constant joint entropy, this entropy criterion can be simplified into a sensing quality function

$$F_H(\mathcal{A}) = H(\mathbf{Y}_{\mathcal{A}}) \quad (3)$$

where $\mathcal{A} \subseteq \mathcal{V}$. Even though this set function can be easily proved to be submodular, for random vectors on continuous probability space the entropy criterion cannot be guaranteed to be non-decreasing, as was noted in [2]. Fundamentally, this is related to problems with differential entropy, which is an improper generalization from discrete to continuous random variables, as was stated by Jaynes [4]. In that sense it is more reasonable to use mutual information that captures the expected information gain that sensing provides for random vectors.

The mutual information criterion can be given with two equal forms

$$\begin{aligned} F_{MI}(\mathcal{A}) &= H(\mathbf{Y}_{\mathcal{V}}) - H(\mathbf{Y}_{\mathcal{V}}|\mathbf{Y}_{\mathcal{A}}) \\ &= H(\mathbf{Y}_{\mathcal{A}}) - H(\mathbf{Y}_{\mathcal{A}}|\mathbf{Y}_{\mathcal{V}}). \end{aligned} \quad (4)$$

and the increment that selection of a new element provides is

$$\begin{aligned} \delta_{MI}(s|\mathcal{A}) &= F_{MI}(\mathcal{A} \cup \{s\}) - F_{MI}(\mathcal{A}) \\ &= H(\mathbf{Y}_{\mathcal{A}}, \mathbf{Y}_s) - H(\mathbf{Y}_{\mathcal{A}}, \mathbf{Y}_s|\mathbf{Y}_{\mathcal{V}}) \\ &\quad - H(\mathbf{Y}_{\mathcal{A}}) + H(\mathbf{Y}_{\mathcal{A}}|\mathbf{Y}_{\mathcal{V}}) \\ &= H(\mathbf{Y}_s|\mathbf{Y}_{\mathcal{A}}) - H(\mathbf{Y}_s|\mathbf{Y}_{\mathcal{A}}, \mathbf{Y}_{\mathcal{V}}). \end{aligned}$$

which is always nonnegative. However, the problem related to near-optimality is that this criterion cannot be considered submodular, since $\delta_{MI}(s|\mathcal{A}) - \delta_{MI}(s|\mathcal{B})$ cannot be guaranteed to be nonnegative.

In sensor placement applications, one is often mostly interested in minimizing the uncertainty of a random vector at sensing sites where these sensors are not placed. This is reasonable, since placing fixed sensors in some sites is expected to provide maximum information about these sites. In their studies, Krause and Guestrin proved that by using the mutual information criterion, this concept provides near-optimal solutions. Now the mutual information criterion is given by

$$F_{MI}(\mathcal{A}) = H(\mathbf{Y}_{\mathcal{V} \setminus \mathcal{A}}) - H(\mathbf{Y}_{\mathcal{V} \setminus \mathcal{A}}|\mathbf{Y}_{\mathcal{A}}) \quad (5)$$

and the increase by

$$\delta_{MI}(s|\mathcal{A}) = H(\mathbf{Y}_s|\mathbf{Y}_{\mathcal{A}}) - H(\mathbf{Y}_s|\mathbf{Y}_{\bar{\mathcal{A}}}) \quad (6)$$

where $\bar{\mathcal{A}} = \mathcal{V} \setminus (\mathcal{A} \cup \{s\})$. This mutual information criterion is submodular, since

$$H(\mathbf{Y}_s|\mathbf{Y}_{\mathcal{A}}) \geq H(\mathbf{Y}_s|\mathbf{Y}_{\mathcal{B}}) \quad \text{where } \mathcal{A} \subseteq \mathcal{B} \quad (7)$$

and

$$H(\mathbf{Y}_s|\mathbf{Y}_{\bar{\mathcal{B}}}) \geq H(\mathbf{Y}_s|\mathbf{Y}_{\bar{\mathcal{A}}}) \quad \text{where } \bar{\mathcal{B}} \subseteq \bar{\mathcal{A}} \quad (8)$$

and non-decreasing when $H(\mathbf{Y}_s|\mathbf{Y}_{\mathcal{A}}) \geq H(\mathbf{Y}_s|\mathbf{Y}_{\bar{\mathcal{A}}})$. In the following subsection we extend this criterion to applications where sensing sites must be visited frequently.

B. Criterion for replicated measurements

Consider a normal vector given by

$$\mathbf{Y}_{\mathcal{V}} = \mathbf{F}_{\mathcal{V}} + \mathbf{W}_{\mathcal{V}} \quad (9)$$

consisting of two independent normal vectors; $\mathbf{F}_{\mathcal{V}}$ relates to a spatial variable and $\mathbf{W}_{\mathcal{V}}$ to noise. Now we define a set of possible sensing sites by

$$\mathcal{V} = \{s_1, \dots, s_m\} \quad (10)$$

and a set of replicated sensing sites by

$$\mathcal{U} = \{s_1^1, \dots, s_1^n, \dots, s_m^1, \dots, s_m^n\} \quad (11)$$

where s_i^j denotes an element at site i with a counting number j . Next, we prove that in this context when $n \rightarrow \infty$, the mutual information criterion is given by

$$F_{MI}(\mathcal{A}) = H(\mathbf{F}_{\mathcal{V}}) - H(\mathbf{F}_{\mathcal{V}}|\mathbf{Y}_{\mathcal{A}}), \quad (12)$$

and when $\mathcal{A} \subset \mathcal{U}$, this is a non-decreasing submodular set function.

Proposition 2.1: Let \mathcal{V} and \mathcal{U} be as given in (10) and (11), respectively. When $n \rightarrow \infty$, mutual information is given by (12).

Proof: Consider mutual information $F_{MI}(\mathcal{A})$ so that

$$\begin{aligned} F_{MI}(\mathcal{A}) &= H(\mathbf{Y}_{\mathcal{A}}) - H(\mathbf{Y}_{\mathcal{A}}|\mathbf{Y}_{\mathcal{U}}) \\ &= H(\mathbf{Y}_{\mathcal{A}}) - \int p(\mathbf{y}_{\mathcal{U}}) H(\mathbf{Y}_{\mathcal{A}}|\mathbf{y}_{\mathcal{U}}) d\mathbf{y}_{\mathcal{U}} \end{aligned} \quad (13)$$

where

$$H(\mathbf{Y}_{\mathcal{A}}|\mathbf{y}_{\mathcal{U}}) = - \int p(\mathbf{y}_{\mathcal{A}}|\mathbf{y}_{\mathcal{U}}) \log p(\mathbf{y}_{\mathcal{A}}|\mathbf{y}_{\mathcal{U}}) d\mathbf{y}_{\mathcal{A}}.$$

From the consistency of a normal vector, for an estimator $\bar{\mathbf{f}}$ we have asymptotic density

$$\lim_{n \rightarrow \infty} p(\bar{\mathbf{f}}_{\mathcal{V}}|\mathbf{y}_{\mathcal{U}}) = \delta(\bar{\mathbf{f}}_{\mathcal{V}} - \mathbf{f}_{\mathcal{V}})$$

where $\delta(\cdot)$ is Dirac's delta function and $\mathbf{f}_{\mathcal{V}}$ is the true realization from which noisy samples $\mathbf{y}_{\mathcal{U}}$ are induced. Now, for conditional density $p(\mathbf{y}_{\mathcal{A}}|\mathbf{y}_{\mathcal{U}})$ we can integrate

$$\begin{aligned} \lim_{n \rightarrow \infty} p(\mathbf{y}_{\mathcal{A}}|\mathbf{y}_{\mathcal{U}}) &= \lim_{n \rightarrow \infty} \int p(\mathbf{y}_{\mathcal{A}}|\bar{\mathbf{f}}_{\mathcal{V}}, \mathbf{y}_{\mathcal{U}}) p(\bar{\mathbf{f}}_{\mathcal{V}}|\mathbf{y}_{\mathcal{U}}) d\bar{\mathbf{f}}_{\mathcal{V}} \\ &= p(\mathbf{y}_{\mathcal{A}}|\mathbf{f}_{\mathcal{V}}, \mathbf{y}_{\mathcal{U}}) \\ &= p(\mathbf{y}_{\mathcal{A}}|\mathbf{f}_{\mathcal{V}}) \end{aligned}$$

where the last line follows from conditional independency. Finally, from the invariance of a probability measure (in the differential) we have

$$p(\mathbf{y}_{\mathcal{U}}) d\mathbf{y}_{\mathcal{U}} = p(\mathbf{f}_{\mathcal{V}}) d\mathbf{f}_{\mathcal{V}}$$

Substituting these into equation (13), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} F_{MI}(\mathcal{A}) &= H(\mathbf{Y}_{\mathcal{A}}) - H(\mathbf{Y}_{\mathcal{A}}|\mathbf{F}_{\mathcal{V}}) \\ &= H(\mathbf{F}_{\mathcal{V}}) - H(\mathbf{F}_{\mathcal{V}}|\mathbf{Y}_{\mathcal{A}}) \end{aligned}$$

Proposition 2.2: When $\mathcal{A} \subset \mathcal{U}$, mutual information given by (12) is a non-decreasing submodular set function. ■

Proof: For the mutual information, the increment is given by

$$\begin{aligned}\delta_{MI}(s|\mathcal{A}) &= H(\mathbf{Y}_s, \mathbf{Y}_{\mathcal{A}}) - H(\mathbf{Y}_s, \mathbf{Y}_{\mathcal{A}}|\mathbf{F}_{\mathcal{V}}) \\ &- H(\mathbf{Y}_{\mathcal{A}}) + H(\mathbf{Y}_{\mathcal{A}}|\mathbf{F}_{\mathcal{V}}) \\ &= H(\mathbf{Y}_s|\mathbf{Y}_{\mathcal{A}}) - H(\mathbf{Y}_s|\mathbf{Y}_{\mathcal{A}}, \mathbf{F}_{\mathcal{V}}),\end{aligned}\quad (14)$$

and when $\mathcal{A} \subset \mathcal{U}$, the mutual information increment reduces to

$$\delta_{MI}(s|\mathcal{A}) = H(\mathbf{Y}_s|\mathbf{Y}_{\mathcal{A}}) - H(\mathbf{Y}_s|\mathbf{F}_{\mathcal{V}}) \quad (15)$$

where the last part follows from consistency, i.e., $\mathbf{f}_{\mathcal{V}}$ contains all possible information from basis \mathcal{V} . This criterion is non-decreasing and also submodular, since

$$\begin{aligned}\delta_{MI}(s|\mathcal{A}) - \delta_{MI}(s|\mathcal{B}) &= H(\mathbf{Y}_s|\mathbf{Y}_{\mathcal{A}}) - H(\mathbf{Y}_s|\mathbf{Y}_{\mathcal{B}}) \\ &\geq 0\end{aligned}$$

where $\mathcal{A} \subseteq \mathcal{B} \subset \mathcal{U}$. \blacksquare

From purely a modeling perspective, the proposed criterion is reasonable, since it describes the expected information gain for a spatial model, given noisy measurements. In autonomous sensing we are interested in extending this to continuous sensing spaces, which is presented in the following subsection.

C. Sensing in continuous metric spaces

Consider a continuous metric sensing space given by $\mathcal{X} \subset \mathbf{R}^d$ where d denotes the dimension of a sensing area. Instead of random vectors, randomness in metric spaces is expressed by stochastic processes and in relation to normal distributions, by Gaussian processes. A Gaussian process (cf. [10]) is defined by

$$f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}), k(\mathbf{x}, \hat{\mathbf{x}})) \quad (16)$$

where $m(\mathbf{x})$ is the mean function and $k(\mathbf{x}, \hat{\mathbf{x}})$ is the covariance function. Now the set of all replicated sensing sites is given by $\mathcal{S} = \mathbf{X}^n$ and, in principle, we could now formulate a near-optimal sensing application in continuous metric space using criterion (12). However, in practice this is intractable, and instead we are now interested in knowing the expected information gain that some arbitrarily selected set of sensing sites $\mathcal{A} \subset \mathbf{X}^k$, $|\mathcal{A}| = k$ would provide for a Gaussian process marginalized on basis \mathcal{V} .¹ In our tests with different valid covariance functions, it seemed that the optimal sensing sites could be, in fact, within sites \mathcal{V} . Based on those tests, we propose the following conjecture.

Conjecture 2.3: Let $\mathcal{A} \subset \mathbf{X}^k$, $|\mathcal{A}| = k$, and \mathcal{V} and \mathcal{U} are given by (10), (11), respectively. Now the maximum mutual information $\max F_{MI}(\mathcal{A})$, where $F_{MI}(\mathcal{A})$ is given by (12), is provided when $\mathcal{A} \subset \mathcal{U}$.

The premises needed to prove the conjecture are based on the entropy of normal distributions and the positive definiteness of a covariance function. Since Proposition 2.1 did not restrict \mathcal{A} to belong to any special set, (12) is now

¹Note that \mathcal{A} is a discrete metric subspace of \mathbf{X}^k .

also a valid presentation of sensing quality. For a Gaussian process marginalized in \mathcal{V} , entropy is given by

$$\begin{aligned}H(\mathbf{F}_{\mathcal{V}}) &= \log \sqrt{(2\pi e)^m |K_{\mathcal{V}\mathcal{V}}|} \\ &= 0.5 \log |K_{\mathcal{V}\mathcal{V}}| + \text{const.}\end{aligned}\quad (17)$$

where $K_{\mathcal{V}\mathcal{V}}$ is the covariance matrix among vector elements. Respectively, conditional entropy is given by

$$\begin{aligned}H(\mathbf{F}_{\mathcal{V}}|\mathbf{Y}_{\mathcal{A}}) &= \int 0.5 \log |K_{\mathcal{V}\mathcal{V}|\mathbf{y}_{\mathcal{A}}}| p(\mathbf{y}_{\mathcal{A}}) d\mathbf{y}_{\mathcal{A}} + \text{const.} \\ &= 0.5 \log |K_{\mathcal{V}\mathcal{V}|\mathcal{A}}| + \text{const.}\end{aligned}\quad (18)$$

where the last line follows when we state that the hyperparameters are fixed. For predictive variance we have

$$K_{\mathcal{V}\mathcal{V}|\mathcal{A}} = K_{\mathcal{V}\mathcal{V}} - K_{\mathcal{V}\mathcal{A}}^T (K_{\mathcal{A}\mathcal{A}} + R_{\mathcal{A}})^{-1} K_{\mathcal{V}\mathcal{A}} \quad (19)$$

where $R_{\mathcal{A}}$ is a diagonal noise variance matrix. In that sense the mutual information criterion depends only on $K_{\mathcal{V}\mathcal{A}}^T (K_{\mathcal{A}\mathcal{A}} + R_{\mathcal{A}})^{-1} K_{\mathcal{V}\mathcal{A}}$, and it is reasonable to believe that this is maximized when $\mathcal{A} \subset \mathcal{U}$.

III. SLAM EXPLORATION

In the SLAM framework, the modeling problem is related to the uncertainty of sensing sites, which can be stated as $(\mathbf{X}^k, \Sigma, \mu)$ so that μ is a probability measure $p(\mathbf{x}^k) d\mathbf{x}^k$. In autonomous modeling this uncertainty can be described by a stochastic process $\mathbf{X}^k = (\mathbf{X}_1, \dots, \mathbf{X}_k)$ ² which is usually assumed to be first-order Markovian, so that,

$$\mathbf{X}_{i+1} = g(\mathbf{X}_i, \mathbf{U}_i) \quad (20)$$

where $g(\mathbf{X}_i, \mathbf{U}_i)$ is called a motion model and \mathbf{U}_i is a random control variable. In this context, sensing quality can be described as an action selection problem, so that,

$$\mathbb{E}[F_{MI}(\mathbf{a})] = \int F_{MI}(\mathbf{x}^k(\mathbf{a})) p(\mathbf{x}^k(\mathbf{a})) d\mathbf{x}^k(\mathbf{a}), \quad (21)$$

where

$$\mathbf{X}^k(\mathbf{a}) = (\mathbf{X}_1, \mathbf{X}_2(a_1), \dots, \mathbf{X}_k(a_{k-1})),$$

$\mathbf{a} = (a_1, \dots, a_{k-1})$ and $\mathbf{U}_i = \mathbf{U}(a_i)$. Now we are trying to find a sequence of actions $\mathbf{a} \in \mathbf{A}^{k-1}$ for which the expected mutual information given by (21) is maximized. For submodular functions, any convex combination of submodular functions is also a submodular function. In that sense, the expected mutual information is also a submodular function.

In principle, in each iteration we could use a greedy algorithm to find the optimal action $a_i^* \in \mathbf{A}$. However, this wouldn't necessarily guarantee near-optimality unless action space \mathbf{A} is continuous, since we would be required to attain all possible combinations of sensing sites, which in practice is intractable. Instead, we rely on Conjecture 2.3 to find near-optimal actions when sensing sites are given by particle discretization. This is presented in the following subsection.

²Note that \mathbf{X}^k refers to a sample space and \mathbf{X}^k to a random vector $\mathbf{X}^k : \mathbf{X}^k \rightarrow \mathbf{X}^k$.

A. Action selection policy

Consider a discrete distribution of sensing sites so that sensing quality is given by

$$\mathbb{E}[F_{MI}(\mathbf{a})] = \sum_{i=1}^q F_{MI}(\mathbf{x}^k(\mathbf{a})) P(\mathbf{x}^k(\mathbf{a})) \quad (22)$$

where $\sum_{i=1}^q P(\mathbf{x}^k(\mathbf{a})) = 1$ and q is the number of particles. Instead of searching through all possible actions, here we base our approach on Conjecture 2.3. We now require for all particles

$$\begin{aligned} F_{MI}(\mathbf{x}^k(\mathbf{a}))_{greedy} &\geq (1 - \alpha)F_{MI}(\mathcal{A})_{greedy} \\ &\geq (1 - \alpha)(1 - e^{-1})F_{MI}(\mathcal{A})_{OPT} \end{aligned}$$

where $\mathcal{A} \subset \mathcal{U}$. If Conjecture 2.3 is true, this guarantees near-optimality over the continuous metric sensing space \mathbf{X}^k . If Conjecture 2.3 is not true, this still guarantees near-optimality compared with the optimal solution $F_{MI}(\mathcal{A})_{OPT}$ at discrete sensing sites \mathcal{V} . In practice this means that, on each iteration, we can select an action that provides a near-optimal increment for each particle drawn from the motion model. In order to make this practically faster, we compute actions required to reach from the current expected site to sites that provide near-optimal solutions for an ideal case, i.e., when the sensing site distribution is given by a delta function. From these actions we select only those that provide near-optimality for all particles.

So far, we have considered actions that provide near-optimality for modeling. In SLAM exploration, we are interested in actions that provide minimum posterior uncertainty for sensing sites while guaranteeing near-optimality for modelling. In the following subsection we develop equations to quantify uncertainties related to sensing sites in Gaussian processes.

B. Minimizing the uncertainty of sensing sites

Following particle discretization, we can build a Gaussian approximation for sensing sites so that

$$\mathbf{X}^k(\mathbf{a}) \sim \mathcal{N}(\boldsymbol{\mu}_k(\mathbf{a}), \mathbf{C}_{kk}(\mathbf{a})) \quad (23)$$

where $\boldsymbol{\mu}_k(\mathbf{a})$ and $\mathbf{C}_{kk}(\mathbf{a})$ are weighted sample means and covariance matrix, respectively, for actions $\mathbf{a} = (a_1, \dots, a_{k-1})$. Particle weights can be computed using either marginal likelihood over all measurements, LOO predictive probability among measurements, or simply by iteratively multiplying the previous weight value with the likelihood of the latest measurement. In our experiments, we did not find a significant difference between these approaches, and we used the last alternative on our implementations.

In order to estimate the posterior uncertainties of sensing sites for each possible action $a \in \mathbf{A}$, we need to linearize the Gaussian process with a first-order Taylor approximation. In our approach, we used a Taylor approximation around a mean function, i.e.,

$$m(\mathbf{x}_{k+1}) = m(\boldsymbol{\mu}_{k+1}) + m'(\boldsymbol{\mu}_{k+1})(\mathbf{x}_{k+1} - \boldsymbol{\mu}_{k+1}). \quad (24)$$

After applying k measurements we have a predictive mean on the expected sensing sites, given by

$$\begin{aligned} \overline{m(\boldsymbol{\mu}_{k+1})} &= m(\boldsymbol{\mu}_{k+1}) + \mathbf{k}_{k+1,k}^T (\mathbf{K}_{k,k} + \mathbf{R}_{k,k})^{-1} \\ &\cdot (\mathbf{y}_k - m(\boldsymbol{\mu}_k)) \end{aligned} \quad (25)$$

where $\mathbf{k}_{k+1,k}$ is a cross-covariance vector among the predictive and measurement sites and $\mathbf{K}_{k,k}$ is a covariance matrix for the measurement sites.

Jacobian w.r.t. $k+1$ sites and dimensions of the sites is given by

$$\mathbf{H} = \left(\frac{\partial \overline{m(\boldsymbol{\mu}_{k+1})}}{\partial x_{1,1}}, \dots, \frac{\partial \overline{m(\boldsymbol{\mu}_{k+1})}}{\partial x_{1,d}}, \dots, \frac{\partial \overline{m(\boldsymbol{\mu}_{k+1})}}{\partial x_{k+1,d}} \right)$$

where for $i = 1, \dots, k$ sites, partial derivatives are given by

$$\begin{aligned} \frac{\partial \overline{m(\boldsymbol{\mu}_{k+1})}}{\partial x_{i,j}} &= \mathbf{k}_{k+1,k}^T (\mathbf{K}_{k,k} + \mathbf{R}_{k,k})^{-1} \left(-\frac{\partial m(\boldsymbol{\mu}_k)}{\partial x_{i,j}} \right) \\ &+ \frac{\partial \mathbf{k}_{k+1,k}^T}{\partial x_{i,j}} \cdot (\mathbf{K}_{k,k} + \mathbf{R}_{k,k})^{-1} \\ &\cdot (\mathbf{y}_k - m(\boldsymbol{\mu}_k)) \\ &- \mathbf{k}_{k+1,k} \cdot (\mathbf{K}_{k,k} + \mathbf{R}_{k,k})^{-1} \\ &\cdot \frac{\partial \mathbf{K}_{k,k}}{\partial x_{i,j}} (\mathbf{K}_{k,k} + \mathbf{R}_{k,k})^{-1} \\ &\cdot (\mathbf{y}_k - m(\boldsymbol{\mu}_k)) \end{aligned}$$

and for the $k+1$ th site, by

$$\begin{aligned} \frac{\partial \overline{m(\boldsymbol{\mu}_{k+1})}}{\partial x_{k+1,j}} &= \frac{\partial m(\boldsymbol{\mu}_{k+1})}{\partial x_{k+1,j}} \\ &+ \frac{\partial \mathbf{k}_{k+1,k}^T}{\partial x_{k+1,j}} \cdot (\mathbf{K}_{k,k} + \mathbf{R}_{k,k})^{-1} \\ &\cdot (\mathbf{y}_k - m(\boldsymbol{\mu}_k)). \end{aligned}$$

In order to quantify sensing site uncertainty, we can use the differential entropy of normal random vectors given by

$$H(\mathbf{X}^{k+1} | \mathbf{y}_k) = 0.5 \log |C_{k+1,k+1} |_{\mathbf{y}_k} | + \text{const.}$$

where covariance $C_{k+1,k+1} |_{\mathbf{y}_k}$ is computed using the extended Kalman filter update. In each iteration we now select the near-optimal action for which the entropy of the sensing sites is minimized. The full iterative procedure is presented in Algorithm 1.

At the input we have the set of sensing sites \mathcal{V} , particle distribution \mathbf{X}^k , greedy increase $\delta^* = \max_{s \in \mathcal{V}} \delta_{MI}(s | \mathcal{A})$, where $|\mathcal{A}| = k$, error boundary α , and measurements \mathbf{y}_k . At the output we have the near-optimal action a^* and the updated particle distribution \mathbf{X}^{k+1} . On lines 2 - 7 we build a set of actions that provide near-optimal solutions for an ideal sensing site distribution. On lines 8 - 15 we restrict this to only those actions that are guaranteed to provide near-optimality for all particles. On line 16 we select the action that provides minimum posterior uncertainty for the sensing sites. Finally, on line 17 we update each particle with a sensing site provided by the near-optimal action a^* .

Algorithm 1 Near-optimal SLAM exploration iteration

Input: \mathcal{V} , \mathbf{X}^k , δ^* , α , \mathbf{y}_k
Output: \mathbf{X}^{k+1} , a^*

- 1: $\mathbf{A} \leftarrow \emptyset$
- 2: **for all** $s \in \mathcal{V}$ **do**
- 3: **if** $\delta(s|E(\mathbf{X}^k)) \geq (1 - \alpha)\delta^*$ **then**
- 4: compute control action a_s from $E(\mathbf{X}^k)$ to s
- 5: $\mathbf{A} \leftarrow \mathbf{A} \cup \{a_s\}$
- 6: **end if**
- 7: **end for**
- 8: **for all** $a \in \mathbf{A}$ **do**
- 9: **for all** $\mathbf{x}^k \in \mathbf{X}^k$ **do**
- 10: draw $\mathbf{x}_{k+1}(a) \sim g(\mathbf{x}_k, \mathbf{U}(a))$
- 11: **end for**
- 12: **if** $\exists \mathbf{x}^k \in \mathbf{X}^k : \delta(\mathbf{x}_{k+1}(a)|\mathbf{x}_k) < (1 - \alpha)\delta^*$ **then**
- 13: $\mathbf{A} \leftarrow \mathbf{A} \setminus \{a\}$
- 14: **end if**
- 15: **end for**
- 16: $a^* \leftarrow \operatorname{argmin}_{a \in \mathbf{A}} H(\mathbf{X}^k, \mathbf{X}_{k+1}(a)|\mathbf{y}_k)$
- 17: $\mathbf{X}^{k+1} \leftarrow \mathbf{X}^k \cup \mathbf{X}_{k+1}(a^*)$

IV. EXPERIMENTS

In order to provide some preliminary quantification, we built three simulations that considered near-optimal SLAM exploration in stationary Gaussian processes with known parameters and hyperparameters. In these simulations we used three-dimensional vector fields modeled by three orthogonal Gaussian processes. The reason for this is in our magnetic field localization studies [3], where we used 3-axis magnetometers.

In the first two simulations we considered a somewhat easier concept where the mean function is a function of the sensing sites. This can be viewed as an example of SLAM exploration in the areas where we have learned the parametrical model based on previous measurements in some discrete sensing sites. In the third simulation we used a constant mean function, which is more practical in totally unknown areas, however harder to deal with in localization. The quality of the models was evaluated by their mean squared errors, whereas for sensing site uncertainties we used only a graphical representation. The following subsections present the simulation results with required details of the models.

A. Simulation models

The sensor model for our magnetometers is given by

$$\mathbf{Z}^3(\mathbf{x}, \theta) = \mathbf{R}(\theta)\mathbf{Y}^3(\mathbf{x}) \quad (26)$$

where $\mathbf{R}(\theta)$ is a left-handed rotation matrix. This is indeed an ideal case, since in order to cope with sensor inclination and non-orthogonality in the sensor axis, we used an additional calibration matrix in the actual experiments. Following from the sensor model, mean function partial derivatives must be computed also for orientation. We won't go into the details here; however, in practical use it was easier to

compute these w.r.t. $\cos(\theta)$ and $\sin(\theta)$. Finally, all Jacobian elements are transformed into sensor coordinates, so that

$$\mathbf{H}_Z^3 = (\mathbf{H}_Z^3(x_{1,1}), \mathbf{H}_Z^3(x_{1,2}), \mathbf{H}_Z^3(\cos(\theta_1)), \mathbf{H}_Z^3(\sin(\theta_1)), \dots, \mathbf{H}_Z^3(x_{k+1,1}), \mathbf{H}_Z^3(x_{k+1,2}), \mathbf{C}_Z^3, \mathbf{S}_Z^3),$$

where

$$\mathbf{H}_Z^3(\cdot) = \mathbf{R}^{-1}(\theta_{k+1})\mathbf{H}^3(\cdot)$$

$$\mathbf{C}_Z^3 = (\overline{m_1(\boldsymbol{\mu}_{k+1})}, \overline{m_2(\boldsymbol{\mu}_{k+1})}, 0)^T$$

$$\mathbf{S}_Z^3 = (-\overline{m_2(\boldsymbol{\mu}_{k+1})}, \overline{m_1(\boldsymbol{\mu}_{k+1})}, 0)^T$$

and \mathbf{C}_Z^3 , \mathbf{S}_Z^3 are partial derivative vectors w.r.t. $\cos(\theta_{k+1})$ and $\sin(\theta_{k+1})$, respectively.

For Gaussian processes we used a squared exponential covariance function given by

$$k(\mathbf{x}, \hat{\mathbf{x}}) = \sigma_f^2 \cdot \exp\left(-\frac{|\mathbf{x} - \hat{\mathbf{x}}|^2}{l^2}\right)$$

where σ_f^2 is signal variance and l^2 the length-scale hyperparameter, and the partial derivative is now given by

$$\frac{\partial k(\mathbf{x}_i, \mathbf{x}_k)}{\partial x_{i,j}} = -2 \cdot (x_{i,j} - x_{k,j}) \cdot k(\mathbf{x}_i, \mathbf{x}_k).$$

In our simulations we used $\sigma_f^2 = 5^2$ and $l^2 = (500\text{mm})^2$ for all models, whereas our sensing environment was restricted to a square area with lower and upper corners $(0\text{mm}, 0\text{mm})$, $(4000\text{mm}, 4000\text{mm})$, respectively.

For each vector field component we drew a dataset of size $(41,41)$, presented in Fig. 7, from a zero mean model with the hyperparameters given above. In the simulations these were summed with mean functions, where we used the same mean function for each process. Three different mean functions are given by

$$m(\mathbf{x}) = 20 \cdot \sin(0.001 \cdot x_1) + 20 \cdot \sin(0.001 \cdot x_2),$$

$$m(\mathbf{x}) = 0.01 \cdot x_1 + 0.01 \cdot x_2, \quad m(\mathbf{x}) = 0$$

and noisy samples were generated by using bilinear interpolation and noise $\sigma_n^2 = 0.1^2$.

The motion model used in the simulations is given by

$$\begin{pmatrix} X_{i+1,1} \\ X_{i+1,2} \\ \Theta_{i+1} \end{pmatrix} = \begin{pmatrix} X_{i,1} + \Delta_t(a_i) \cdot \cos(\Theta_i + \Delta_r(a_i)) \\ X_{i,2} + \Delta_t(a_i) \cdot \sin(\Theta_i + \Delta_r(a_i)) \\ \Theta_i + \Delta_r(a_i) \end{pmatrix}$$

where Δ_t , Δ_r are normally distributed univariate random control variables given by

$$\Delta_r(a) \sim \mathcal{N}(\mu_r(a), \mu_r(a)^2 \cdot \phi_1 + \mu_t(a)^2 \cdot \phi_2)$$

and

$$\Delta_t(a) \sim \mathcal{N}(\mu_t(a), \mu_t(a)^2 \cdot \phi_3 + \mu_r(a)^2 \cdot \phi_4)$$

where the motion model parameters ϕ_1, \dots, ϕ_4 define the variances of these distributions (see e.g. [14]). In the simulations we used

$$\begin{aligned} \phi_1 &= 0.08^2, & \phi_2 &= \left(0.00005 \frac{\text{rad}}{\text{mm}}\right)^2, \\ \phi_3 &= (0.05)^2, & \phi_4 &= \left(8 \frac{\text{mm}}{\text{rad}}\right)^2. \end{aligned}$$

which were intended to model the true error parameters for our robotic platforms.

B. Results

In our simulations we used a setup where the error boundary was set to $\alpha = 0.3$, the number of particles was limited to 50, resampling was performed when the effective sample size was below half of the number of particles, and the set of discrete sensing sites \mathcal{V} consisted of 256 evenly spaced sites.

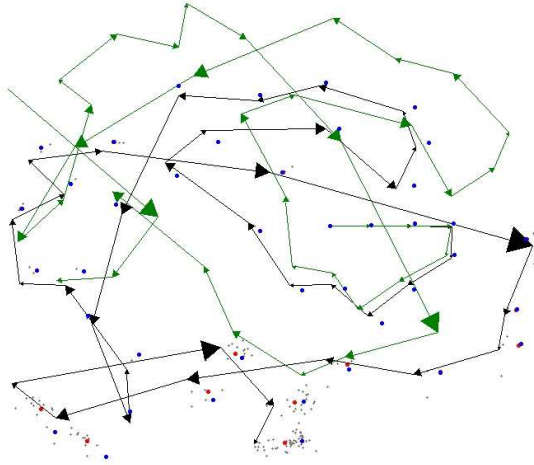


Fig. 1. Particle distribution after the 40th iteration of a sine mean function. Black and green arrows are the true and mean control (odometry) paths, respectively. Grey dots represent particles, red dots expected sensing sites, and blue dots are the most likely sensing sites.

Figures 1, 2 and 3 present typical particle distributions for each simulation model. For the non-constant mean functions we were able to build around 40 iterations until the particle distribution was unable to converge, which caused the algorithm to not find any near-optimal action. As one would expect, for the constant mean function particle dispersion was much faster, which typically allowed algorithm to run only around 20 iterations.

Figures 4, 5, and 6 present the mean squared errors of each simulation and each vector component. These errors were evaluated at sensing sites \mathcal{V} using predictive means provided by the ideal greedy sampler and the near-optimal SLAM explorer w.r.t. the true realizations presented in Fig. 7. In order to compute predictive means for the near-optimal SLAM explorer, we used expected sensing sites as inputs to the Gaussian processes, with necessary rotations for measurements.

These simulations were only preliminary; however, they gave important information about the difficulty involved

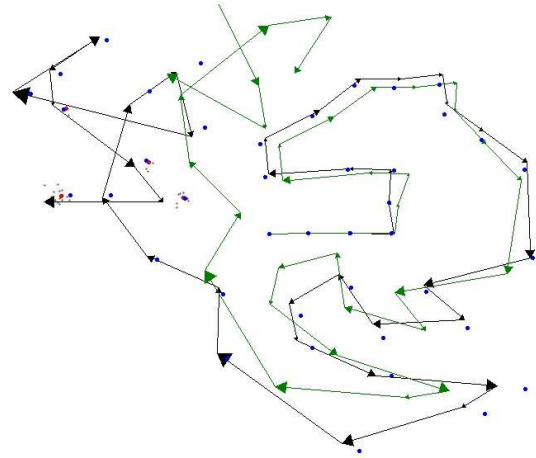


Fig. 2. Particle distribution after the 40th iteration of a linear mean function.

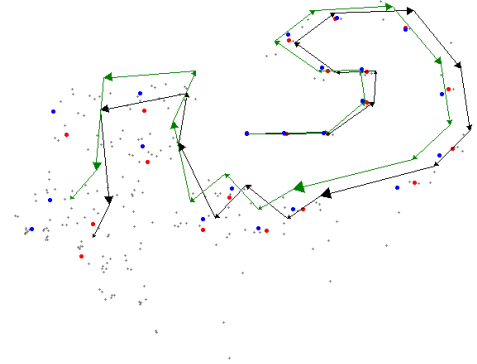


Fig. 3. Particle distribution after the 20th iteration of a constant mean function.

when working in a near-optimal domain. The authors believe that with a greater number of particles, a larger action selection space, and a scalable error boundary, we can extend this to real-world applications using magnetic fields.

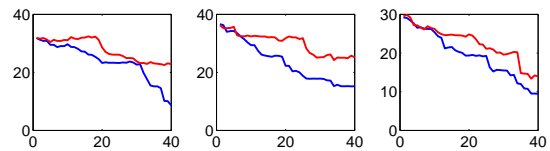


Fig. 4. Mean squared errors w.r.t. sampling iteration of a sine mean function. Red color represents MSE for the near-optimal SLAM explorer and blue color, for the ideal greedy sampler.

V. CONCLUSIONS

In this paper we examined near-optimal SLAM exploration in Gaussian processes. We proved that for applications where sensing sites must be visited frequently, mutual information

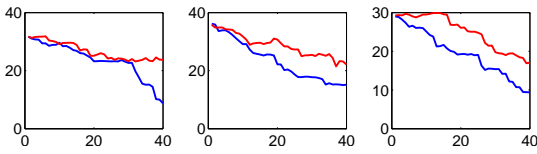


Fig. 5. Mean squared errors w.r.t. sampling iteration of a linear mean function.

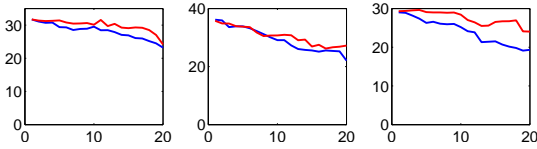


Fig. 6. Mean squared errors w.r.t. sampling iteration of a constant mean function.

provides near-optimal solutions. We extended this to continuous probability spaces where the sensing sites themselves are random variables. We showed that with particle discretization, this framework can attain near-optimality.

In order to provide some preliminary results, we built three simulations of vector fields. These simulations show the difficulty involved when working with a fixed error boundary in near-optimality, even with ideal models. In our magnetic field research domain, these indoor vector fields are highly non-stationary (cf. [3]) which motivates us to use adaptive models, such as presented in [5], in our future work.

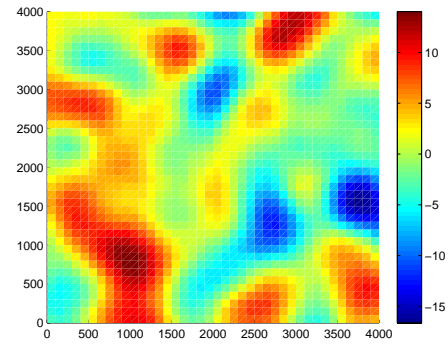
In our parallel experimental paper [15] we conducted experiments with magnetic field SLAM. In that paper we showed that magnetic field SLAM is feasible, given a sufficient amount of replicated measurements and a large particle set. This motivates us to continue our research on near-optimal exploration.

ACKNOWLEDGMENT

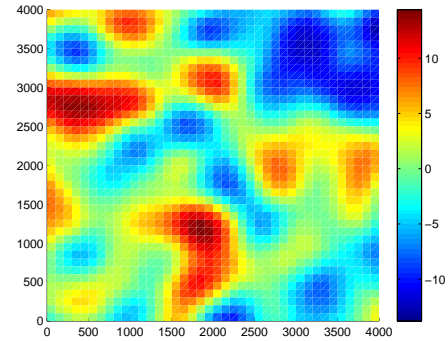
We acknowledge Infotech Oulu and the Academy of Finland for supporting our studies.

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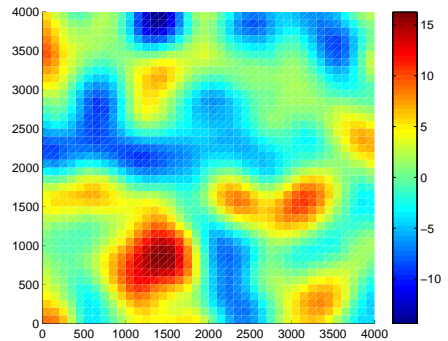
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(a)



(b)



(c)

Fig. 7. Vector field realization (a), (b) and (c) for components f_1 , f_2 and f_3 , respectively.

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